Solution of assignment 3

1 (a) $f(z)=e^{-z}$ analytic in $\bar{D}$, which is the closed domain enclosed by the square

$$\int_C \frac{e^{-z}}{z-\frac{\pi i}{2}}dz = f\left(\frac{\pi i}{2}\right)\cdot 2\pi i = 2\pi$$

(b) $f(z)=\dfrac{\cos z}{z^2+8}$ analytic in $\bar{D}$

$$\int_C \frac{f(z)}{z-0}dz = f(0)\cdot 2\pi i = \frac{\pi i}{4}$$

2. As $n \leq 0$ $\dfrac{e^z}{z^n}$ is analytic in the complex plane

So $\displaystyle\int_C \frac{e^z}{z^n} = 0$

As $n>0$, assume $f(z)=e^z$. By theorem in Note 3

we have $\displaystyle 1=f^{(n-1)}(0)=\frac{(n-1)!}{2\pi i}\int_C \frac{e^z}{z^n}dz$, so

$$\int_C \frac{e^z}{z^n} = \frac{2\pi i}{(n-1)!}$$

3 (a) $\displaystyle f(z)=\sum_{k=0}^{+\infty}\frac{f^{(k)}(0)}{k!}z^k$. Because $f^{(n)}(z)=0$, we have $f^{(m)}(z)=0$

$\forall\, m \geq n$. So $\displaystyle f(z)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}z^k$ which is a polynomial

(b) $\displaystyle f^{(k)}(0)=\frac{k!}{2\pi i}\int_C \frac{f(z)}{z^{k+1}}dz$, where C is a circle of radius

$r>R$, positive oriental.

For any $k \geq n+1$ $\displaystyle |f^{(k)}(0)| = \left|\frac{k!}{2\pi i}\int_C \frac{f(z)}{z^{k+1}}dz\right| \leq \frac{k!}{2\pi}\cdot r^n\cdot\frac{1}{r^{k+1}}\cdot 2\pi r$

let $r \to +\infty$, we have $\frac{k!}{2\pi} \cdot 2\pi \cdot \frac{r^{n+1}}{r^{k+1}} \to 0$ ($\because k+1 > n+2$)

so we have $f^{(k)}(0) = 0$.

Because $f(z)$ already have the representation $\sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!} z^k$,

$\Rightarrow f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$ which is a polynomial.

4. $\int_\gamma \frac{1}{z}(z+\frac{1}{z})^{2n} dz$

$= \int_0^{2\pi} e^{-it}(e^{it}+e^{-it})^{2n} \cdot e^{it} \cdot i\, dt = \underline{\int_0^{2\pi} i(e^{it}+e^{-it})^{2n} dt}$

$= i\int_0^{2\pi} \sum_{k=0}^{2n} \binom{2n}{k} (e^{it})^k (e^{-it})^{2n-k} dt = i\int_0^{2\pi} \sum_{k=0}^{2n} \binom{2n}{k} e^{it(2k-2n)} dt$

$= i\int_0^{2\pi} \binom{2n}{n} dt = \underline{2\pi i \binom{2n}{n}}$

Thus $\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t\, dt = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{e^{it}+e^{-it}}{2}\right)^{2n} dt$

$= \frac{1}{2\pi} \frac{1}{2^{2n}} \int_0^{2\pi} (e^{it}+e^{-it})^{2n} dt = \frac{1}{2\pi} \cdot \frac{1}{2^{2n}} \cdot 2\pi \cdot \binom{2n}{n}$

$= \frac{(2n)!}{(n! \cdot 2^n)^2} = \frac{(2n)!}{(2n \cdot (2n-2) \cdots (2))^2} = \frac{(2n-1)\cdot(2n-3)\cdots 1}{2n \cdot (2n-2) \cdots 2}$

5 $f^{(n)}(z) = e \quad \forall n > 0$

Because e^z is entire, $\Rightarrow e^z = \sum_{k=0}^{+\infty} \frac{e}{k!}(z-1)^k$

6. when $z \neq 0$ $\frac{e^z - 1}{z} = \sum_{k=1}^{+\infty} \frac{z^k}{k!} / z = \sum_{k=1}^{+\infty} \frac{z^{k-1}}{k!} = \sum_{k=0}^{+\infty} \frac{z^k}{(k+1)!}$

Define function $g(z) = \sum_{k=0}^{+\infty} \frac{z^k}{(k+1)!}$ $z \in \mathbb{C}$.

$\Rightarrow g(0) = 1 \Rightarrow g(z) = f(z)$ $\because g(z)$ is an entire function, so is $f(z)$.